Resit exam — Analysis (WPMA14004)

Thursday 9 July 2015, 9.00h–12.00h

University of Groningen

Instructions

1. The use of calculators, books, or notes is not allowed.
2. Provide clear arguments for all your answers: only answering "yes", "no", or "42" is not sufficient. You may use all theorems and statements in the book, but you should clearly indicate which of them you are using.
3. The total score for all questions equals 90. If p is the number of marks then the exam grade is $G = 1 + p/10$.

Problem 1 (6 + 9 points)

Assume that $A \subset \mathbb{R}$ is nonempty and bounded above.

(a) State the definition of "least upper bound of A".

(b) Prove that $-A \stackrel{\text{def}}{=} \{-a \,:\, a \in A\}$ is bounded below and $\inf(-A) = -\sup A$.

Problem 2 (8 + 4 + 3 points)

Assume that $a_n \neq 0$ for all $n \in \mathbb{N}$ and $L = \lim \left|\dfrac{a_{n+1}}{a_n}\right|$ exists. Prove the following statements:

(a) For all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$(L-\epsilon)^k |a_N| < |a_{N+k}| < (L+\epsilon)^k |a_N| \quad \text{for all } k \in \mathbb{N}.$$

(b) $L < 1 \;\Rightarrow\; \sum_{n=1}^{\infty} a_n$ converges absolutely.

(c) $L > 1 \;\Rightarrow\; \sum_{n=1}^{\infty} a_n$ diverges.

Problem 3 (10 + 5 points)

Let $K_n \subset \mathbb{R}$ be compact for all $n \in \mathbb{N}$.

(a) Prove that $\bigcap_{n=1}^{\infty} K_n$ is compact.

(b) Give an example of compact sets $K_n \subset \mathbb{R}$ such that $\bigcup_{n=1}^{\infty} K_n$ is **not** compact.

Problem 4 (10 + 5 points)

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and periodic with period $T > 0$:

$$f(x + T) = f(x) \quad \text{for all } x \in \mathbb{R}.$$

(a) Prove that there exists a constant $M > 0$ such that $|f(x)| \leq M$ for all $x \in \mathbb{R}$.

(b) Assume in addition that $f(x + \frac{1}{2}T) = -f(x)$ for all $x \in \mathbb{R}$. Prove that $f(x) = 0$ for infinitely many points x.

Problem 5 (15 points)

Prove that $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{\log(n + x) - \log(n)}{n}$ is differentiable on $[0, 1]$.

Problem 6 (3 + 12 points)

Define the function $f : [0, 1] \to \mathbb{R}$ as

$$f(x) = \begin{cases} \dfrac{1}{p} & \text{if } x \in \left[\dfrac{p-1}{p}, \dfrac{p}{p+1}\right) \text{ for some } p \in \mathbb{N}, \\ 0 & \text{if } x = 1. \end{cases}$$

(a) Sketch the graph of f on the interval $[0, \frac{3}{4})$.

(b) Prove that f is integrable on $[0, 1]$.
Hint: use partitions for which all subintervals have equal length.

End of test (90 points)

Solution of Problem 1 (6 + 9 points)

(a) A number $s \in \mathbb{R}$ is the least upper bound of A if:

(i) s is an upper bound of A, which means that $a \leq s$ for all $a \in A$;
(3 points)

(ii) if s' is another upper bound for A, then $s \leq s'$.
(3 points)

(b) By the Axiom of Completeness $s = \sup A$ exists. Since $s = \sup A$ is an upper bound of A we have that $a \leq s$ for all $a \in A$. This implies that $-s \leq -a$ for all $a \in A$, which shows that $-s$ is a lower bound of $-A$. We conclude that $-A$ is bounded below.
(4 points)

Since $s = \sup A$ is the *least* upper bound of A we have that for all $\epsilon > 0$ there exists an $a \in A$ such that $s - \epsilon < a$, which is equivalent to $-a < -s + \epsilon$. This proves that for any $\epsilon > 0$ the number $-s + \epsilon$ is no longer a lower bound for $-A$.
(4 points)

We conclude that $-s$ is the *greatest* lower bound for $-A$, i.e., $\inf(-A) = -\sup A$.
(1 point)

Solution of Problem 2 (8 + 4 + 3 points)

(a) Let $\epsilon > 0$ be arbitrary, then there exists $N \in \mathbb{N}$ such that

$$n \geq N \quad \Rightarrow \quad \left| \left| \frac{a_{n+1}}{a_n} \right| - L \right| < \epsilon$$

Rewriting the inequality gives

$$n \geq N \quad \Rightarrow \quad (L - \epsilon)|a_n| < |a_{n+1}| < (L + \epsilon)|a_n|.$$

(4 points)

Setting $n = N$ proves the desired statement for $k = 1$. If the statement is true for some $k \in \mathbb{N}$, then it follows that

$$N+k+1 > N \quad \Rightarrow \quad |a_{N+k+1}| < (L+\epsilon)|a_{N+k}| < (L+\epsilon)(L+\epsilon)^k|a_N| = (L+\epsilon)^{k+1}|a_N|,$$

which proves the inequality for $k + 1$. The other inequality is proven similarly. By induction, the statement holds for all $k \in \mathbb{N}$.
(4 points)

(b) If $L < 1$ we can take $0 < \epsilon < 1 - L$ so that $0 < L + \epsilon < 1$. By part (a) it follows that for n sufficiently large, the terms $|a_n|$ are bounded by the terms of a convergent geometric series. By the Comparison Test it follows that $\sum_{n=N}^{\infty} |a_n|$ converges, which means that $\sum_{n=N}^{\infty} a_n$ converges absolutely.
(4 points)

(c) If $L > 1$ we can take $0 < \epsilon < L - 1$ so that $L - \epsilon > 1$. By part (a) it follows that the sequence (a_n) is unbounded. Therefore, $\sum_{n=1}^{\infty} a_n$ diverges.
(3 points)

Solution of Problem 3 (10 + 5 points)

(a) Let $K = \bigcap_{n=1}^{\infty} K_n$. To prove that K is compact we need to show that K is closed and bounded.
(2 points)

Since K_1 is compact it is bounded. In addition, we have that $K \subset K_1$. This proves that also K is bounded.
(4 points)

Each K_n is closed, because it is compact. The arbitrary intersection of closed sets is again closed. This proves that K is closed as well.
(4 points)

(b) In general the union $\bigcup_{n=1}^{\infty} K_n$ is not compact, even when all sets K_n are compact. For example, each set $K_n = [-n, n]$ is closed and bounded and therefore compact. However, the union

$$\bigcup_{n=1}^{\infty} K_n = \mathbb{R}$$

is unbounded and therefore not compact.
(5 points)

Problem 4 (10 + 5 points)

(a) Since f is continuous on $\mathbb{R}$ also $|f|$ is continuous on $\mathbb{R}$. The set $[0, T]$ is compact and therefore $|f|$ attains a maximum and a minimum on $[0, T]$. We conclude there exists a constant $M > 0$ such that $|f(x)| \leq M$ for all $x \in [0, T]$.
(6 points)

Let $x \in \mathbb{R}$ be arbitrary. Then there exists a number $k \in \mathbb{Z}$ such that $x + kT \in [0, T]$. Since f is T-periodic it follows that

$$|f(x)| = |f(x + kT)| \leq M.$$

(4 points)

(b) If $f(0) = 0$ then the statement is obvious by the periodicity of f.

If $f(0) \neq 0$, then $f(0)$ and $f(\frac{1}{2}T)$ have opposite sign. Since f is continuous we can apply the Intermediate Value Theorem: there exists $c \in (0, \frac{1}{2}T)$ such that $f(c) = 0$.
(4 points)

By periodicity of f we have that $f(c + kT) = 0$ for all $k \in \mathbb{Z}$. We conclude that $f(x) = 0$ for infinitely many points $x \in \mathbb{R}$.
(1 point)

Solution of Problem 5 (15 points)

For all $x \in [0, 1]$ we have that

$$f_n(x) = \frac{\log(n + x) - \log(n)}{n} \quad \Rightarrow \quad f_n'(x) = \frac{1}{n(n + x)} \quad \Rightarrow \quad |f_n'(x)| \leq \frac{1}{n^2}$$

The Weierstrass M-test with $M_n = 1/n^2$ implies that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges} \quad \Rightarrow \quad \sum_{n=1}^{\infty} f_n'(x) \text{ converges uniformly on } [0, 1]$$

(5 points)

Since $f_n(0) = 0$ for all $n \in \mathbb{N}$ the series $\sum_{n=1}^{\infty} f_n(x)$ trivially converges for $x = 0$.

(3 points)

Recall Theorem 6.4.3. Let f_n be differentiable functions on $[a, b]$ and assume that $\sum_{n=1}^{\infty} f_n'(x)$ converges uniformly to some function $g(x)$ on $[a, b]$. If there exists a point $x_0 \in [a, b]$ such that $\sum_{n=1}^{\infty} f_n(x_0)$ converges, then the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly to a differentiable function $f(x)$ on $[a, b]$ with $f'(x) = g(x)$.

(5 points)

Above we have checked that the conditions of the theorem are satisfied. We conclude that the given series defines a differentiable function on $[0, 1]$.

(2 points)

Problem 6 (3 + 12 points)

(a) Note that f is piecewise constant:

$$\begin{aligned} x \in [0, \tfrac{1}{2}) \quad &\Rightarrow \quad f(x) = 1 \\ x \in [\tfrac{1}{2}, \tfrac{2}{3}) \quad &\Rightarrow \quad f(x) = \tfrac{1}{2} \\ x \in [\tfrac{2}{3}, \tfrac{3}{4}) \quad &\Rightarrow \quad f(x) = \tfrac{1}{3} \end{aligned}$$

(1 point for each correct line segment)

(b) Let $P = \{0 = x_0 < x_1 < \cdots < x_n = 1\}$ be a partition of $[0, 1]$ such that

$$x_k - x_{k-1} = \frac{1}{n} \qquad \text{for all } k = 1, \ldots, n.$$

From part (a) it follows that f is decreasing. Therefore, we have that

$$M_k = \sup\{f(x) \,:\, x \in [x_{k-1}, x_k]\} = f(x_{k-1})$$

$$m_k = \inf\{f(x) \,:\, x \in [x_{k-1}, x_k]\} = f(x_k)$$

for all $k = 1, \ldots, n$.

(4 points)

The difference between the upper and lower sum of f with respect to P is

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1}) \\ &= \frac{1}{n} \sum_{k=1}^{n} (f(x_{k-1}) - f(x_k)) \\ &= \frac{f(x_0) - f(x_n)}{n} \\ &= \frac{1}{n}. \end{aligned}$$

(6 points)

By the Archimedean Property of $\mathbb{R}$ there exists $n \in \mathbb{N}$ such that

$$\frac{1}{n} < \epsilon$$

In this case $U(f, P) - L(f, P) < \epsilon$. We conclude that f is integrable on $[0, 1]$.

(2 points)